

# Detecting ill posed boundary conditions in General Relativity

Gioel Calabrese and Olivier Sarbach

*Department of Physics and Astronomy, Louisiana State University,  
202 Nicholson Hall, Baton Rouge, Louisiana 70803-4001*

A persistent challenge in numerical relativity is the correct specification of boundary conditions. In this work we consider a many parameter family of symmetric hyperbolic initial-boundary value formulations for the linearized Einstein equations and analyze its well posedness using the Laplace-Fourier technique. By using this technique ill posed modes can be detected and thus a necessary condition for well posedness is provided. We focus on the following types of boundary conditions: i) Boundary conditions that have been shown to preserve the constraints, ii) boundary conditions that result from setting the ingoing constraint characteristic fields to zero and iii) boundary conditions that result from considering the projection of Einstein's equations along the normal to the boundary surface. While we show that in case i) there are no ill posed modes, our analysis reveals that, unless the parameters in the formulation are chosen with care, there exist ill posed constraint violating modes in the remaining cases.

## I. INTRODUCTION

Obtaining a long time convergent numerical simulation of a binary black hole spacetime in domains with artificial boundaries continues to be a challenge in numerical relativity and one which has recently received a substantial amount of attention, notably in the case of hyperbolic formulations (see [1, 2] for reviews). The challenge remains in part because of the difficulty in specifying boundary conditions. It has been recognized [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] that the boundary conditions have to satisfy two important requirements. First, they have to preserve the constraints. By this we mean that they must guarantee that if the constraints are satisfied initially they are also satisfied at later times. We refer to boundary conditions that satisfy this property as constraint preserving boundary conditions (CPBC). Second, the boundary conditions have to be such that the resulting initial-boundary value problem (IBVP) is well posed. This means that given initial and boundary data a unique solution exists and that at each fixed time the solution depends continuously on the data. Well posedness is a necessary condition for the construction of consistent and stable finite difference schemes [13, 14].

When the evolution equations are in symmetric hyperbolic form one usually specifies maximal dissipative boundary conditions [15]. Under certain technical assumptions, these conditions guarantee that the resulting IBVP is well posed [16, 17]. Using maximal dissipative boundary conditions, Friedrich and Nagy [3] were able to find well posed CPBC for a particular formulation of the full nonlinear vacuum equations. However, most of the hyperbolic formulations used in numerical relativity are based on evolution equations that use a different set of variables than in Ref. [3]. For these formulations, the derivation of well posed CPBC seems to be more difficult. Part of the problem stems from the fact that CPBC result in a set of partial differential equations that must hold at the boundary surface, and it is not always possible to cast these equations into the form of maximal dissipative boundary conditions. This is probably the reason why current well posed CPBC for formulations other than that used in Ref. [3] are either limited to homogeneous boundary data [9] or to linearizations around a Minkowski background [7, 8, 9]. Even in those cases, the CPBC obtained so far might be too restrictive in the sense that they do not allow the specification of the physical quantities at the boundary with the freedom one would like to have. For example, the well posed boundary conditions obtained in Ref. [7] involve a coupling between the in- and outgoing variables and, likely, this coupling will introduce reflections at the boundary. Therefore, more general techniques are desirable in order to show well posedness for more generic CPBC.

In this article, we use the Laplace-Fourier technique to analyze boundary conditions in linearized General Relativity. This technique is very useful when the evolution equations are linear and have constant coefficients since it can be applied to boundary conditions that are more general than the maximal dissipative ones. Specifically, it can be applied to boundary conditions which have the form of differential equations at the boundary. Furthermore, the method is capable of detecting the presence of ill posed modes analytically. Ill posed modes are solutions to the IBVP that grow exponentially in time with an exponential factor that can be arbitrarily large, and their existence makes it impossible for the solution to depend continuously on the data. The Laplace-Fourier technique therefore provides us with a necessary condition for well posedness. However, it should be emphasized that the absence of ill posed modes (as defined in this article) does not automatically guarantee well posedness. Although more complicated in this case, results for the variable coefficient case are available by freezing the coefficients at the boundary (see [18, 19]).

This article is organized as follows: The conditions under which the specification of non-maximal dissipative boundary conditions for symmetric hyperbolic systems with constant coefficients yields ill posed modes are reviewed in section

II. In section III we discuss the boundary conditions that have been considered for the generalized Einstein-Christoffel formulation of Einstein's equations [20, 21] when linearized around flat spacetime. The generalized Einstein-Christoffel system is a family of symmetric hyperbolic formulations that is parametrized by a constant  $\eta$ . The boundary conditions we are considering are: i) The CPBC that were considered in Ref. [7] and that are based on solving a closed evolution system at the boundary and on maximal dissipative boundary conditions. ii) Boundary conditions that are obtained by setting the ingoing constraint characteristic fields to zero. iii) Boundary conditions that are obtained by considering the projection of Einstein's equations along the normal to the boundary surface, as recently proposed by Frittelli and Gomez [10, 11]. In section IV we apply the techniques discussed in section II and show that the cases ii) and iii) suffer from the presence of ill posed modes unless the parameter  $\eta$  in the generalized Einstein-Christoffel formulation lies in a specific range. We also show that there are no ill posed modes in case i) which is consistent with the well posedness estimates derived in Ref. [7]. In section V we show that the ill posed modes we have found in cases ii) and iii) do, in fact, all violate the constraints. This means that the evolution system for the constraint variables is ill posed in those cases. Since this system is always strongly hyperbolic and since our boundary conditions are constructed from specifying maximal dissipative conditions for this system, this also illustrates that maximal dissipative boundary conditions do not necessarily yield a well posed formulation if the evolution equations are strongly hyperbolic (but not symmetrizable). In particular, our calculations show that the boundary conditions that are constructed following the schemes ii) and iii) do not necessarily lead to CPBC and that one should always check the evolution system for the constraint variables. Our results and their implications on deriving well posed CPBC are discussed in section VI. A similar analysis for the Frittelli-Reula [22] system has been undertaken by Stewart [4].

## II. DETECTING ILL POSED MODES

In this section, we review the techniques that can reveal the presence of ill posed modes. They are based on a Laplace transformation in time and on a Fourier transformation in the spatial directions that are tangential to the boundaries and are described in more detail in Refs. [13, 23]. For simplicity, we restrict the following discussion to the 2D case; the generalization to 3D is straightforward.

Consider a 2D first order in time and space linear evolution equation of the form

$$\partial_t u = A \partial_x u + B \partial_y u, \quad (1)$$

where  $u = u(t, x, y)$  is a vector-valued function and the matrices  $A$  and  $B$  are constant and symmetric. We consider solutions to Eq. (1) on the domain  $t > 0$ ,  $x > 0$ ,  $-\pi < y < +\pi$  with initial data

$$u(0, x, y) = f(x, y) \quad (2)$$

and boundary conditions at  $x = 0$  of the form

$$L(\partial_t, \partial_y)u(t, 0, y) = g(t, y), \quad (3)$$

where  $L$  is a linear operator with constant coefficients that only involves derivatives which are tangential to the boundary. For technical reasons, we assume that  $L(\partial_t, \partial_y)$  is homogeneous in the sense that  $L(\mu \partial_t, \mu \partial_y) = \mu L(\partial_t, \partial_y)$  for all positive  $\mu$ . We also assume periodic boundary conditions in the  $y$ -direction (similar conclusions hold for the case  $-\infty < y < +\infty$ ).

The IBVP (1), (2), (3) is said to be well posed [24], if given smooth square integrable data  $f, g$  there exists a unique smooth solution. Furthermore, there are constants  $C, a$  such that

$$\|u(t, \cdot)\|^2 \leq C e^{at} \left[ \|f\|^2 + \int_0^t \|g(\tau, \cdot)\|^2 d\tau \right], \quad (4)$$

for all  $t > 0$  and all square integrable data  $f$  and  $g$ . Here,  $\|u(t, \cdot)\|$  denotes the  $L^2$  norm of  $u$  defined by  $\|u(t, \cdot)\|^2 = \int_{x>0} |u(t, x, y)|^2 dx dy$  and similarly,  $\|f\|^2 = \int_{x>0} |f(x, y)|^2 dx dy$  and  $\|g(\tau, \cdot)\|^2 = \int |g(\tau, y)|^2 dy$ . The estimate (4) implies that for each fixed  $t$ , the solution depends continuously on the data  $f$  and  $g$ .

A first step in checking if a given initial-boundary formulation satisfies a well posedness inequality of the type (4) is to look for solutions of the problem with homogeneous data ( $g = 0$ ) which are of the form

$$u(t, x, y) = e^{st+i\omega y} \tilde{u}(x), \quad (5)$$

where  $\omega$  is an integer,  $s$  is complex with  $\text{Re}(s) > 0$  and  $\tilde{u}(x)$  is a smooth function that lies in  $L^2(0, \infty)$ . If such a solution exists, the problem cannot be well posed. In order to see this we notice that the functions

$$u_m(t, x, y) = e^{m(st+i\omega y)} \tilde{u}(mx), \quad (6)$$

where  $m = 1, 2, 3, \dots$  can be arbitrarily large are also solutions and since  $\|u_m(t, \cdot)\|/\|u_m(0, \cdot)\| = \exp(m\operatorname{Re}(s)t)$  the estimate (4) cannot hold with constants  $C$  and  $a$  that are independent of the initial data. Therefore, an obvious check for well posedness is to see whether or not Eqs. (1), (3) admit nontrivial solutions of the form (5) with homogeneous boundary data.

Using expression (5) in Eqs. (1), (3), we obtain (for  $g = 0$ )

$$s\tilde{u} = A\partial_x\tilde{u} + i\omega B\tilde{u}, \quad (7)$$

$$L(s, i\omega)\tilde{u}(0) = 0. \quad (8)$$

These equations form a system of ordinary differential equations and can be solved analytically. In order to do so, we first bring  $A$  to diagonal form through an orthonormal transformation. Thus, the matrix  $B$  is still symmetric and we can write

$$A = \begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix},$$

where  $A_1 = \operatorname{diag}(\lambda_1, \dots, \lambda_p, \lambda_{p+1}, \dots, \lambda_{p+q})$  with  $\lambda_1, \dots, \lambda_p$  real and negative and  $\lambda_{p+1}, \dots, \lambda_{p+q}$  real and positive. Here,  $p$  and  $q$  are the number of in- and outgoing modes, respectively. Accordingly, we write

$$B = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix}, \quad L = (L_0, L_1), \quad \tilde{u} = \begin{pmatrix} \tilde{u}_0 \\ \tilde{u}_1 \end{pmatrix}.$$

Now the zero components of Eq. (7) yield the following algebraic relation between  $\tilde{u}_0$  and  $\tilde{u}_1$ :

$$S_{00}\tilde{u}_0 = -S_{01}\tilde{u}_1, \quad (9)$$

where we have introduced the matrix  $S = S(s, \omega) = sI - i\omega B$ . Since the matrix  $B$  is symmetric, the matrix  $S_{00} = sI - i\omega B_{00}$  is invertible for all  $\operatorname{Re}(s) > 0$  and all integer  $\omega$ , and we can express  $\tilde{u}_0$  in terms of  $\tilde{u}_1$ :

$$\tilde{u}_0 = -S_{00}^{-1}S_{01}\tilde{u}_1. \quad (10)$$

Inserting this into the remaining equations of the system (7), (8), we obtain the reduced problem

$$\partial_x\tilde{u}_1 = M(s, \omega)\tilde{u}_1, \quad (11)$$

$$\tilde{L}\tilde{u}_1 = 0, \quad (12)$$

where

$$\begin{aligned} M(s, \omega) &= A_1^{-1} (S_{11}(s, \omega) - S_{10}S_{00}^{-1}S_{01}(s, \omega)), \\ \tilde{L}(s, \omega) &= L_1 - L_0S_{00}^{-1}S_{01}. \end{aligned}$$

One can show [19] that for  $\operatorname{Re}(s) > 0$  the matrix  $M(s, \omega)$  has exactly  $p$  eigenvalues with negative real parts and exactly  $q$  eigenvalues with positive real parts (the eigenvalues are counted according to their algebraic multiplicity.)

The eigenvalues of  $M$  that have positive real part lead to exponential growth in  $x$ . Since the solution  $\tilde{u}$  has to be in  $L^2(0, \infty)$ , the integration constants have to be chosen such that there is no such growth in  $x$ . In order to achieve this, we choose, for each  $(s, \omega)$ , a unitary matrix  $U = U(s, \omega)$  that brings  $M(s, \omega)$  in upper triangular form:

$$U(s, \omega)^{-1}M(s, \omega)U(s, \omega) = \begin{pmatrix} M_-(s, \omega) & M_0(s, \omega) \\ 0 & M_+(s, \omega) \end{pmatrix}. \quad (13)$$

Here,  $M_-$  ( $M_+$ ) is an upper triangular matrix whose eigenvalues have negative (positive) real parts. If we introduce the new variable  $v(x) = U(s, \omega)^{-1}\tilde{u}_1(x)$ , system (11), (12) becomes

$$\begin{aligned} \partial_x v_-(x) &= M_-(s, \omega)v_-(x) + M_0(s, \omega)v_+(x), \\ \partial_x v_+(x) &= M_+(s, \omega)v_+(x), \\ L_-v_-(0) + L_+v_+(0) &= 0, \end{aligned}$$

where  $(L_-, L_+) = \tilde{L}U$ . It follows that  $v_+$  must vanish for  $v$  to be in  $L^2(0, \infty)$ . This implies that  $v_-(x) = \exp(M_-x)\sigma_-$  where  $\sigma_-$  has to satisfy the boundary condition  $L_-\sigma_- = 0$ . We conclude that the system (7,8) has only the trivial solution if and only if the determinant condition[25]

$$\det L_-(s, \omega) \neq 0, \quad \operatorname{Re}(s) > 0, \quad (14)$$

is satisfied. (In particular,  $L_-$  must be a square matrix of dimension  $p$ . This means that we need exactly as many independent boundary conditions as there are ingoing modes). If the determinant condition is violated at some point  $(s, \omega) = (s_0, \omega_0)$ , it is also violated for  $(s, \omega) = m(s_0, \omega_0)$  with  $m = 1, 2, 3, \dots$  and the initial-boundary formulation admits solutions of the form (5) that grow exponentially in time where the exponential factor  $s$  can have arbitrarily large real part.

In section IV we will discuss the determinant condition for the case of the linearized Einstein equations with boundaries.

### III. BOUNDARY CONDITIONS FOR THE LINEARIZED EINSTEIN-CHRISTOFFEL SYSTEM

In this section we discuss boundary conditions for a linearization of the generalized Einstein-Christoffel vacuum equations [20]. This formulation has the attractive feature that when linearized around flat spacetime written in Minkowski coordinates it simply reduces to a set of six wave equations, written in first order form:

$$\partial_t K_{ij} = -\delta^{kl} \partial_k f_{lij}, \quad (15)$$

$$\partial_t f_{kij} = -\partial_k K_{ij}. \quad (16)$$

Here,  $K_{ij}$  denotes the linearized extrinsic curvature and the symbols  $f_{kij}$  represent linear combinations of the linearized Christoffel symbols  $\Gamma_{kij}$ :

$$f_{kij} = \Gamma_{(ij)k} + \delta^{rs} \left( \delta_{ki} \Gamma_{[sj]r} + \delta_{kj} \Gamma_{[si]r} + \frac{\eta - 4}{2\eta} \delta_{ij} \Gamma_{[sk]r} \right). \quad (17)$$

The value of  $\eta$  (which has to be different from zero) parametrizes the family of formulations. The particular case with  $\eta = 4$  corresponds to the original Einstein-Christoffel system derived by Anderson and York [21]. We set the shift to zero, and the lapse is linearized in such a way that it satisfies the densitized lapse gauge condition  $\alpha = \sqrt{g}$  up to second order corrections, where  $g$  denotes the determinant of the three metric. A solution to the system (15), (16) is a solution to the linearized Einstein equations if and only if the constraints are satisfied. In terms of the constraint variables

$$C = \frac{\eta}{4} \delta^{rs} \partial_r v_s, \quad (18)$$

$$C_j = \delta^{rs} (\partial_r K_{sj} - \partial_j K_{rs}), \quad (19)$$

$$C_{lkij} = 2\partial_{[l} f_{k]ij} + \eta \partial_{[l} \delta_{k](i} v_{j)} + \frac{\eta - 4}{4} \delta_{ij} \partial_{[l} v_{k]}, \quad (20)$$

where  $v_k = \delta^{ij}(f_{kij} - f_{ijk})$ , the constraints are given by  $C = 0$ ,  $C_j = 0$ ,  $C_{lkij} = 0$ .

We consider the evolution system (15), (16) on the domain  $t > 0$ ,  $x > 0$ ,  $-\pi < y, z < +\pi$  and introduce the characteristic variables in the  $x$  direction[26]

$$u_{ij}^{(-)} = \frac{1}{\sqrt{2}} (K_{ij} + f_{xij}), \quad (21)$$

$$u_{ij}^{(+)} = \frac{1}{\sqrt{2}} (K_{ij} - f_{xij}), \quad (22)$$

$$u_{Aij}^{(0)} = f_{Aij}. \quad (23)$$

Here and in the following capital Latin indices stand for the tangential directions  $y$  and  $z$ . When written in terms of these variables the evolution equations (15), (16) take the form

$$\partial_t u_{ij}^{(-)} = -\partial_x u_{ij}^{(-)} - \frac{1}{\sqrt{2}} \delta^{AB} \partial_A u_{Bij}^{(0)}, \quad (24)$$

$$\partial_t u_{ij}^{(+)} = +\partial_x u_{ij}^{(+)} - \frac{1}{\sqrt{2}} \delta^{AB} \partial_A u_{Bij}^{(0)}, \quad (25)$$

$$\partial_t u_{Aij}^{(0)} = -\frac{1}{\sqrt{2}} \partial_A \left( u_{ij}^{(-)} + u_{ij}^{(+)} \right), \quad (26)$$

and we see that the matrix  $A$  in Eq. (1) is diagonal.

For the constraints to be satisfied everywhere, when boundaries are present, one has to ensure that they are satisfied initially and that no constraint violating mode enters the domain. In order to ensure this, we follow the analysis in Ref. [7] and first consider the evolution of the constraint variables with respect to the flux defined by the main evolution equations (15), (16). One can show [7] that the traceless part of  $C_{lkij}$  is constant in time, while the remaining constraints propagate according to

$$\partial_t C = \frac{\eta}{4} \delta^{ij} \partial_i C_j, \quad (27)$$

$$\partial_t C_j = \frac{4-2\eta}{\eta} \partial_j C - \delta^{rs} \partial_r T_{sj}, \quad (28)$$

$$\partial_t T_{ij} = -\partial_i C_j + \left(1 - \frac{3\eta}{4}\right) \partial_j C_i + \frac{\eta}{4} \delta_{ij} \delta^{rs} \partial_r C_s, \quad (29)$$

$$\partial_t V_{ij} = \left(\frac{7\eta}{4} - 3\right) \partial_{[i} C_{j]}, \quad (30)$$

where  $T_{ij} = \delta^{rs}(C_{rijs} + C_{ijrs})$ , and  $V_{ij} = \delta^{rs} C_{ijrs}$ . Introducing  $\kappa = 1 - 3\eta/4$  and the variables

$$C_{ij} = T_{ij} + \frac{2\eta-4}{\eta} \delta_{ij} C = \delta^{rs} (\partial_r f_{ijs} - \partial_j f_{irs}) + \kappa \partial_j v_i - \kappa \delta_{ij} \delta^{rs} \partial_r v_s, \quad (31)$$

the characteristic fields can be written as [27]

$$V_j^{(-)} = \frac{1}{\sqrt{2}} (C_j + C_{xj}), \quad (32)$$

$$V_j^{(+)} = \frac{1}{\sqrt{2}} (C_j - C_{xj}), \quad (33)$$

$$V_{Aj}^{(0)} = C_{Aj} + \kappa (\delta_{xj} C_{xA} - \delta_{Aj} C_{xx}), \quad (34)$$

$$\tilde{V}_{ij}^{(0)} = -\frac{7\kappa+2}{3} C_{[ij]} + (\kappa+1) V_{ij}. \quad (35)$$

If the system (27), (28), (29), (30) is symmetric (or symmetrizable) hyperbolic, one can guarantee that if the constraints are satisfied initially and if homogeneous maximal dissipative boundary conditions are given, the constraints will be satisfied everywhere. Therefore, we consider boundary conditions at  $x = 0$  which are of the form

$$V_x^{(-)} - a V_x^{(+)} = 0, \quad V_A^{(-)} - b V_A^{(+)} = 0, \quad (36)$$

$$u_{xx}^{(-)} - c u_{xx}^{(+)} = g_{xx}, \quad \hat{u}_{AB}^{(-)} - d \hat{u}_{AB}^{(+)} = \hat{g}_{AB}, \quad (37)$$

where the magnitudes of  $a, b, c$  and  $d$  are smaller or equal to 1 and  $\hat{u}_{AB}^{(\mp)} = u_{AB}^{(\mp)} - \frac{1}{2} \delta_{AB} \delta^{CD} u_{CD}^{(\mp)}$  denotes the traceless part of  $u_{AB}^{(\mp)}$ . In order to express the conditions (36) in terms of the main variables  $K_{ij}, f_{kij}$ , we use the definition of the constraint variables, Eqs. (18), (19), (31), and the evolution equations (15), (16) in order to trade  $x$ -derivatives by time and tangential derivatives:

$$V_x^{(\mp)} = \delta^{AB} \left[ \pm \partial_t u_{AB}^{(\mp)} + \partial_A u_{xB}^{(\mp)} \pm \frac{\delta^{CD}}{\sqrt{2}} \partial_C u_{DAB}^{(0)} \pm \frac{\kappa}{2} \partial_A \left( \sqrt{2} \delta^{CD} u_{CDB}^{(0)} - \sqrt{2} \delta^{ij} u_{Bij}^{(0)} + u_{xB}^{(-)} - u_{xB}^{(+)} \right) \right],$$

$$V_A^{(\mp)} = \mp \partial_t u_{xA}^{(\mp)} + \delta^{CD} \partial_C u_{AD}^{(\mp)} - \delta^{ij} \partial_A u_{ij}^{(\mp)} \mp \frac{\delta^{CD}}{\sqrt{2}} \partial_C u_{DxA}^{(0)} \mp \frac{\kappa}{2} \delta^{CD} \partial_A \left( \sqrt{2} u_{CDx}^{(0)} - u_{CD}^{(-)} + u_{CD}^{(+)} \right).$$

It follows from the energy estimates derived in Ref. [7] that when  $0 < \eta < 2$  the conditions (36) guarantee that the constraints are satisfied everywhere if they are satisfied initially. In the following, we will also consider other values of  $\eta$  and show that one might have ill posed modes if the parameter  $\eta$  lies outside the interval  $(0, 2)$ . Notice that the conditions (36) do not involve derivatives normal to the boundary ( $\partial_x$ ). They can be interpreted as evolution equations for the variables  $\delta^{AB}(u_{AB}^{(-)} + a u_{AB}^{(+)})$  and  $u_{xA}^{(-)} + b u_{xA}^{(+)}$  at the boundary. The functions  $g_{xx}$  and  $\hat{g}_{AB}$  are data that can be given freely for a combination of the in- and outgoing gauge and physical variables, respectively [7].

In the next section, we will analyze the following choices of parameters:

1.  $a = -1, b = 1, c = d = 1$ :

This corresponds to the Neumann boundary conditions that we have discussed in Ref. [7]. In this case the

boundary conditions can be recast in a closed evolution system at the boundary. Its solutions provide boundary data for the main evolution system in the form of maximal dissipative boundary conditions. When  $0 < \eta < 2$  one can derive well posedness estimates for the resulting IBVP and the boundary conditions can indeed be called CPBC.

2.  $a = 1, b = -1, c = d = -1$ :

This corresponds to the Dirichlet conditions specified in Ref. [7]. They can also be recast in a closed evolution system at the boundary, and for  $0 < \eta < 2$  one has a well-posed IBVP with CPBC.

3.  $a = 0, b = 0$ :

This corresponds to setting the ingoing constraint variables to zero and might be the most obvious choice for obtaining CPBC. However, we will show in the next section that the resulting IBVP possesses ill posed modes unless the parameter  $\eta$  is chosen appropriately.

4.  $a = 0, b = 1$ :

These are the conditions that one obtains after linearizing the boundary conditions that were recently proposed in Ref. [11]. There, the Einstein-Christoffel formulation ( $\eta = 4$ ) is considered and the boundary conditions (36) are obtained by projecting Einstein's equations along the normal to the boundary rather than by analyzing the evolution of the constraints. In fact, one can show that setting  $G_{xA}$  to zero and rewriting[28] the resulting equations in terms of the variables  $K_{ij}$  and  $f_{kij}$  is equivalent to the second equation in (36) with  $b = 1$ , while setting  $G_{xx}$  ( $G_{tx}$ ) to zero is equivalent to the first equation in (36) with  $a = 1$  ( $a = -1$ ). In Ref. [11], the authors propose to set the linear combination  $G_{xx} - G_{tx}$  to zero which would correspond to using  $a = 0$  in (36). In the next section, we show that the resulting boundary conditions yield an ill-posed formulation if the parameter  $\eta$  is not chosen appropriately.

#### IV. LAPLACE-FOURIER ANALYSIS

Following the analysis described in section II, we look for solutions to Eqs. (15), (16), (36), (37) with homogeneous boundary data ( $g_{xx} = 0, \hat{g}_{AB} = 0$ ) and which are of the form

$$u_{ij}^{(\mp)}(t, x, y, z) = e^{st + i\omega_y y + i\omega_z z} \tilde{u}_{ij}^{(\mp)}(x), \quad (38)$$

$$u_{Aij}^{(0)}(t, x, y, z) = e^{st + i\omega_y y + i\omega_z z} \tilde{u}_{Aij}^{(0)}(x), \quad (39)$$

where  $s$  is a complex number with positive real part and  $\omega_y$  and  $\omega_z$  are integers. For the solution to be square integrable, we require the functions  $\tilde{u}_{ij}^{(\mp)}(x)$  and  $\tilde{u}_{Aij}^{(0)}(x)$  to be in  $L^2(0, \infty)$ . From Eqs. (38), (39) and (26) we obtain an algebraic condition

$$s\tilde{u}_{Aij}^{(0)} = -\frac{i}{\sqrt{2}}\omega_A \left( \tilde{u}_{ij}^{(-)} + \tilde{u}_{ij}^{(+)} \right) \quad (40)$$

that can be used to eliminate the variable  $\tilde{u}_{Aij}^{(0)}$  from the remaining equations. Inserting Eqs. (38), (39), (40) into Eqs. (24), (25) yields the ordinary differential equation

$$\partial_x \begin{pmatrix} \tilde{u}_{ij}^{(-)} \\ \tilde{u}_{ij}^{(+)} \end{pmatrix} = M(s, \underline{\omega}) \begin{pmatrix} \tilde{u}_{ij}^{(-)} \\ \tilde{u}_{ij}^{(+)} \end{pmatrix}, \quad (41)$$

where

$$M(s, \underline{\omega}) = \begin{pmatrix} -s - \frac{\omega^2}{2s} & -\frac{\omega^2}{2s} \\ \frac{\omega^2}{2s} & s + \frac{\omega^2}{2s} \end{pmatrix} \quad (42)$$

and  $\underline{\omega} = (\omega_y, \omega_z)$ . The matrix  $M(s, \underline{\omega})$  has the eigenvalues  $\pm \sqrt{s^2 + \underline{\omega}^2}$ .

We first look at the case  $\underline{\omega} = 0$ , which corresponds to solutions that have trivial  $y$  and  $z$  dependence. For those, the matrix  $M(s, \underline{\omega})$  is diagonal and since  $\text{Re}(s) > 0$  we see that we must have  $\tilde{u}_{ij}^{(+)} = 0$  for the solution to be in  $L^2$ . The boundary conditions (36), (37) yield

$$\tilde{u}_{xx}^{(-)}(0) = 0, \quad \hat{\tilde{u}}_{AB}^{(-)}(0) = 0, \quad s\delta^{AB}\tilde{u}_{AB}^{(-)}(0) = 0, \quad s\tilde{u}_{xA}^{(-)}(0) = 0, \quad (43)$$

therefore we have only the trivial solution. There are no ill posed modes with trivial dependence on the variables that are tangential to the boundary. We show now that the situation becomes rather more complicated when one considers modes that have a nontrivial tangential dependence.

Assume that  $\underline{\omega} \neq 0$ . Following the analysis of section II we introduce a unitary matrix  $U = U(s, \underline{\omega})$  that brings the matrix  $M(s, \underline{\omega})$  into upper triangular form. To lighten the notation we introduce the quantities  $\zeta = s/|\underline{\omega}|$ ,  $\lambda = \sqrt{1 + \zeta^2}$ ,  $\psi(\zeta) = (\lambda - \zeta)^2$  and  $N = 1 + |\psi(\zeta)|^2$ . One can then verify that the matrix (a star denotes complex conjugation)

$$U(\zeta) = N^{-1/2} \begin{pmatrix} -\frac{|\zeta|}{\zeta} & \frac{|\zeta|}{\zeta^*} \psi^* \\ \frac{|\zeta|}{\zeta} \psi & \frac{|\zeta|}{\zeta^*} \end{pmatrix} \quad (44)$$

is unitary and satisfies

$$U(\zeta)^{-1} M U(\zeta) = |\underline{\omega}| \begin{pmatrix} -\lambda & M_0(\zeta) \\ 0 & \lambda \end{pmatrix}, \quad M_0(\zeta) = \frac{1 + \psi^*(2 + 4\zeta^2 + \psi^*)}{2\zeta^* N}, \quad (45)$$

where in  $\lambda$  the branch is chosen such that for  $\text{Re}(\zeta) > 0$ ,  $\text{Re}(\lambda) > 0$ . In terms of the new variables  $(v_{ij}^{(-)}, v_{ij}^{(+)})^T = U^{-1}(\tilde{u}_{ij}^{(-)}, \tilde{u}_{ij}^{(+)})^T$  Eq. (41) yields

$$|\underline{\omega}|^{-1} \partial_x v_{ij}^{(-)} = -\lambda v_{ij}^{(-)} + M_0(\zeta) v_{ij}^{(+)}, \quad (46)$$

$$|\underline{\omega}|^{-1} \partial_x v_{ij}^{(+)} = \lambda v_{ij}^{(+)}. \quad (47)$$

For the solution to be in  $L^2$ , we must have  $v_{ij}^{(+)} = 0$ . This implies that  $v_{ij}^{(-)} = e^{-\lambda|\underline{\omega}|x} \sigma_{ij}$ , where  $\sigma_{ij}$  are constants which describe the value that  $v_{ij}^{(-)}$  takes at the boundary. Using the matrix  $U(\zeta)$  we can express the  $\tilde{u}$  variables at the boundary as

$$\tilde{u}_{ij}^{(-)}(0) = -N^{-1/2} \frac{|\zeta|}{\zeta} \sigma_{ij}, \quad (48)$$

$$\tilde{u}_{ij}^{(+)}(0) = N^{-1/2} \frac{|\zeta|}{\zeta} \psi(\zeta) \sigma_{ij}, \quad (49)$$

$$\tilde{u}_{Aij}^{(0)}(0) = \frac{i}{\sqrt{2}} \frac{\hat{\omega}_A}{\zeta} N^{-1/2} \frac{|\zeta|}{\zeta} (1 - \psi(\zeta)) \sigma_{ij}, \quad (50)$$

where  $\hat{\omega}_A = \omega_A/|\underline{\omega}|$ .

Using Eqs. (38), (48) and (49) in the boundary condition (37), we find that

$$(1 + c\psi(\zeta))\sigma_{xx} = 0, \quad (1 + d\psi(\zeta))\hat{\sigma}_{AB} = 0, \quad (51)$$

where  $\hat{\sigma}_{AB}$  denotes the tracefree part of  $\sigma_{AB}$ . Since the function  $\psi(\zeta)$  maps the half plane  $\text{Re}(\zeta) > 0$  to the interior of the unit circle and since  $|c| \leq 1$ ,  $|d| \leq 1$ , it follows that  $\sigma_{xx} = 0$  and  $\hat{\sigma}_{AB} = 0$ .

Next, we insert all of this into the boundary conditions (36). The result is more conveniently expressed if one introduces a normalized vector  $\hat{\xi}_A$  that is orthogonal to  $\hat{\omega}_A$  and considers the components  $\sigma_{x\omega} = \delta^{AB} \sigma_{xA} \hat{\omega}_B$  and  $\sigma_{x\xi} = \delta^{AB} \sigma_{xA} \hat{\xi}_B$ . The projection of the second equation in (36) along  $\hat{\xi}$  implies that  $\sigma_{x\xi}$  must vanish, while the remaining equations in (36) imply that  $\sigma \equiv \delta^{AB} \sigma_{AB}$  and  $\sigma_{x\omega}$  must satisfy the following  $2 \times 2$  system:

$$L_-(\zeta) \begin{pmatrix} \sigma \\ \sigma_{x\omega} \end{pmatrix} = 0, \quad L_-(\zeta) = \begin{pmatrix} 2\lambda(1 + a\psi) - \kappa(1 + a)(\lambda - \zeta) & 2i(1 + a\psi) + i\kappa(1 + a)(1 + \psi) \\ i(1 + b\psi) - i\kappa(1 + b)(1 + \psi) & 2\lambda(1 + b\psi) + 2\kappa(1 + b)(\lambda - \zeta) \end{pmatrix}. \quad (52)$$

The determinant of  $L_-(\zeta)$  is given by

$$\det L_-(\zeta) = (6 + 4\zeta^2) [(1 + a\psi(\zeta))(1 + b\psi(\zeta)) - \kappa^2(1 + a)(1 + b)\psi(\zeta)]. \quad (53)$$

Clearly, the first factor cannot be zero since  $\text{Re}(\zeta) > 0$ . Therefore,  $\det L_-(\zeta)$  can only vanish if the term inside the square brackets does.

We now focus on the different cases discussed in the previous section:

1.  $a = -1, b = 1, c = d = 1$ :

In this case, the second term inside the square brackets vanishes and the first term is never zero since  $|\psi(\zeta)| < 1$ . Therefore, the resulting formulation possesses no ill posed modes. Of course, when  $0 < \eta < 2$  this is consistent with our calculation in Ref. [7] where the estimates we have derived exclude the presence of ill posed modes.

2.  $a = 1, b = -1, c = d = -1$ :

The result is the same as in the previous case.

3.  $a = 0, b = 0$ :

In this case, the terms inside the square brackets simplify to  $1 - \kappa^2 \psi(\zeta)$ . A small calculation reveals that this can only be zero if  $\zeta = (\kappa^2 - 1)/2|\kappa|$  and  $\kappa \neq 0$ . Therefore,  $\det L_-(\zeta)$  has a zero with  $\text{Re}(\zeta) > 0$  if and only if  $\kappa^2 > 1$ . This is equivalent to  $\eta < 0$  or  $\eta > 8/3$ . Therefore, setting the ingoing constraint variables to zero in the family of generalized Einstein-Christoffel systems does indeed yield ill posed boundary conditions if the parameter  $\eta$  lies outside the interval  $[0, 8/3]$ .

4.  $a = 0, b = 1$ :

Here the terms inside the square brackets reduce to  $1 + (1 - 2\kappa^2)\psi(\zeta)$ . Since the function  $\psi$  maps the positive real axis onto the open interval  $(0, 1)$  it follows that this expression never vanishes if and only if  $\kappa^2 \leq 1$ . In particular, one has ill posed modes when  $\eta = 4$  and the boundary conditions that were proposed in Ref. [11] yield, at least when linearized around flat spacetime, an ill posed initial-boundary formulation. On the other hand, if at the boundary one considers the equations  $G_{xy} = G_{xz} = 0$  and the equation  $G_{xt} = 0$  instead of the combination  $G_{xx} - G_{xt} = 0$  one has  $a = -1$  and the resulting formulation does not in fact suffer from possessing ill posed modes.

## V. VIOLATIONS OF THE CONSTRAINTS

In this section, we show that the ill posed modes we have found in the previous section violate the constraints. In order to see this, we use these ill posed modes to compute the constraint variables  $C_j$ . From  $K_{ij} = (u_{ij}^{(-)} + u_{ij}^{(+)})/\sqrt{2}$ , Eqs. (19), (48), (49) and  $\sigma_{xx} = 0, \hat{\sigma}_{AB} = 0$ , we have

$$C_x = -\frac{|\underline{\omega}|}{\sqrt{2N}} \frac{|\zeta|}{\zeta} (1 - \psi(\zeta)) (\lambda\sigma + i\sigma_{x\omega}) \exp \left[ |\underline{\omega}| (\zeta t - \lambda x + i\hat{\omega}_A x^A) \right], \quad (54)$$

$$\omega^A C_A = \frac{|\underline{\omega}|}{\sqrt{8N}} \frac{|\zeta|}{\zeta} (1 - \psi(\zeta)) (i\sigma + 2\lambda\sigma_{x\omega}) \exp \left[ |\underline{\omega}| (\zeta t - \lambda x + i\hat{\omega}_A x^A) \right], \quad (55)$$

where  $(\sigma, \sigma_{x\omega})$  is a nontrivial solution to Eq. (52). Since  $|\psi(\zeta)| < 1$  for  $\text{Re}(\zeta) > 0$ , and since

$$\det \begin{pmatrix} \lambda & i \\ i & 2\lambda \end{pmatrix} = 3 + 2\zeta^2 \neq 0, \quad (56)$$

we see that the variables  $C_x$  and  $\omega^A C_A$  cannot simultaneously vanish. Therefore, all the ill posed modes we have found are *constraint violating* modes. This means that under generic small perturbations of the initial data these modes will be excited and the constraint variables will grow exponentially with an exponential factor that can be arbitrarily large. In this sense, the boundary conditions that lead to ill posed modes do *not* preserve the constraints. We point out that the constraint variables constructed from any solution of the main evolution system (15), (16) with boundary conditions (36), (37) provide a solution of the evolution of the constraint variables, Eqs. (27), (28), (29), (30) with boundary conditions (36). Since we have shown that the constraint variables constructed from ill posed modes are ill posed modes themselves (see Eqs. (54), (55)), the IBVP for the constraint variables cannot be well posed. This emphasizes the importance of looking at the evolution system for the constraints and checking its well posedness when deriving CPBC for Einstein's equations.

We conclude this section with two remarks. First, one can check that the evolution system for the constraint variables, Eqs. (27), (28), (29), (30), is strongly hyperbolic for any nonvanishing value of the parameter  $\eta$ . On the other hand, our analysis above shows the existence of ill posed modes when  $\eta$  lies outside of the interval  $[0, 8/3]$  and the coupling constants  $a$  and  $b$  are chosen as in the cases 3. and 4. of the previous section. This illustrates that applying maximal dissipative boundary conditions to evolution systems that are strongly hyperbolic (but not symmetrizable) does not necessarily yield a well posed problem.

The second remark concerns the choice  $a = b = -1$  for the coupling constants in Eq. (36). In this case it follows that the determinant condition is always satisfied, regardless of the value for the parameter  $\eta$ . In fact, one can show



that the resulting boundary conditions are constraint preserving: The evolution equations imply that the constraint variable  $C_j$  satisfies the wave equation:

$$\partial_t^2 C_j = \delta^{rs} \partial_r \partial_s C_j . \quad (57)$$

On the other hand, the choice  $a = b = -1$  corresponds to imposing the momentum constraint at the boundary. Since  $C_j$  satisfies the wave equation, this implies that  $C_j = 0$  everywhere, if  $C_j$  is satisfied initially. It then follows from Eqs. (27), (29), (30) that the remaining constraints are also satisfied if they are satisfied initially. This explains why one has CPBC for all  $\eta \neq 0$  when  $a = b = -1$ . However, the above argumentation is expected to break down when one considers the nonlinear regime since in this case lower order terms might prevent one from obtaining a closed system for  $C_j$  alone. In this case, one has to rely on the symmetrizer for the system (27), (28), (29), (30) which was constructed in Ref. [7], and one might not be able to show that the constraints propagate when  $\eta$  lies outside the interval  $[0, 2)$ , even when  $a = b = -1$ .

## VI. CONCLUSIONS

We have analyzed ill posed modes in the family of the generalized Einstein-Christoffel formulation of Einstein's equations with boundaries. We considered boundary conditions which result from coupling the ingoing characteristic constraint variables to the outgoing ones. Specifically, the cases we have studied include the boundary conditions we have obtained in Ref. [7] and the boundary conditions that originate from considering the projection of Einstein's equations along the normal to the boundary. When linear fluctuations around Minkowski space are considered, we have shown that the formulation is subject to constraint violating ill posed modes unless the parameters in the equations and the coupling between the in- and outgoing constraint variables are chosen carefully. In fact, it is not difficult to show that if the coupling constants  $a$  and  $b$  are real and satisfy  $-1 < a \leq 1$  and  $-1 < b \leq 1$  there are always ill posed modes as long as the parameter  $\eta$  lies outside the interval  $[0, 8/3]$ . In particular, this is the case when the ingoing constraint variables are set to zero. Furthermore, there are ill posed modes for the boundary conditions that were obtained in Ref. [11] when applied to the linearized Einstein-Christoffel system ( $\eta = 4$ ). However, our analysis also reveals that these ill posed modes could easily be avoided by imposing a different linear combination of Einstein's equations at the boundary or by changing the parameter  $\eta$  such that it lies in the interval  $0 < \eta \leq 8/3$ . In any case, our analysis highlights the importance of studying the evolution system for the constraint variables and ensuring its well posedness since all the ill posed modes we have found are constraint violating. In particular, the formulations we have studied in this article show that even though the main evolution system is symmetric hyperbolic, the evolution equations for the constraint variables is not necessarily symmetrizable. For the cases in which the propagation of the constraints is described by a system that is strongly hyperbolic (but not symmetrizable) we have shown that specifying maximal dissipative boundary conditions can lead to an ill posed system.

It is interesting to note that all the ill posed modes that appear have a nontrivial dependence in the spatial directions that are tangential to the boundary surface. Therefore, such modes would not be present in the one-dimensional case. This might explain why the numerical simulations in Ref. [6], where the Einstein-Christoffel system ( $\eta = 4$ ) was evolved using boundary conditions obtained by setting the ingoing constraints to zero, did not show any ill posed modes.

The simple analytic method we have used in this article, which is based on the determinant condition (14), should be used to test the well posedness of the boundary conditions before numerically evolving any evolution system since the presence of ill posed modes would detrimentally affect numerical stability. However, we also stress that more work is required to derive sufficient conditions for well posedness for the choices of parameters when the determinant condition is satisfied. In particular, it would be worthwhile to analyze CPBC where the incoming physical variables can be freely specified.

## Acknowledgments

We particularly thank Gabriel Nagy for many valuable discussions. We also thank L. Lehner, J. Pullin, O. Reula and M. Tiglio for useful comments. This work was supported in part by grants NSF-PHY-9800973, NSF-INT-0204937, and the Horace C. Hearne Jr. Institute of Theoretical Physics.

---

[1] O. Reula, Living Reviews in Relativity **1**, 3 (1998).

- [2] L. Lehner, *Class. Quantum Grav.* **18**, R25 (2001).
- [3] H. Friedrich and G. Nagy, *Comm. Math. Phys.* **201**, 619 (1999).
- [4] J.M. Stewart, *Class. Quantum Grav.* **15**, 2865 (1998).
- [5] M.S. Iriondo and O.A. Reula, *Phys. Rev. D* **65**, 044024 (2002).
- [6] G. Calabrese, L. Lehner, and M. Tiglio, *Phys. Rev. D* **65**, 104031 (2002).
- [7] G. Calabrese, J. Pullin, O. Sarbach, M. Tiglio, and O. Reula, “*Well posed constraint preserving boundary conditions for the linearized Einstein equations*,” arXiv:gr-qc/0209017.
- [8] B. Szilagyi, B. Schmidt, and J. Winicour, *Phys. Rev. D* **65**, 064015 (2002).
- [9] B. Szilagyi and J. Winicour, “*Well Posed Initial-Boundary Evolution in General Relativity*,” arXiv:gr-qc/0205044.
- [10] S. Frittelli and R. Gomez, “*Einstein boundary conditions for the 3+1 Einstein equations*,” arXiv:gr-qc/0302071.
- [11] S. Frittelli and R. Gomez, “*Boundary conditions for hyperbolic formulations of the Einstein equations*,” arXiv:gr-qc/0302032.
- [12] J.M. Bardeen and L.T. Buchman, *Phys. Rev. D* **65**, 064037 (2002).
- [13] B. Gustafsson, H. Kreiss, and J. Olinger, “*Time dependent problems and difference methods*,” John Wiley & Sons, New York (1995).
- [14] G. Calabrese, J. Pullin, O. Sarbach and M. Tiglio, *Phys. Rev. D* **66**, 041501 (2002).
- [15] P.D. Lax, and R.S. Phillips, *Commun. Pure Appl. Math.* **13**, 427 (1960).
- [16] P. Secchi, *Diff. Int. Eq.* **9**, 671 (1996); *Arch. Rat. Mech. Anal.* **134**, 595 (1996).
- [17] J. Rauch, *Trans. Am. Math. Soc.* **291**, 167 (1985).
- [18] H. Kreiss, *Commun. Pure Appl. Math.* **23**, 277 (1970).
- [19] A. Majda and S. Osher, *Commun. Pure Appl. Math.* **28**, 607 (1975).
- [20] L.E. Kidder, M.A. Scheel, and S.A. Teukolsky, *Phys. Rev. D* **64**, 064017 (2001).
- [21] A. Anderson and J.W. York, Jr., *Phys. Rev. Lett.* **82**, 4384 (1999).
- [22] S. Frittelli and O.A. Reula, *Phys. Rev. Lett.* **76**, 4667 (1996).
- [23] H.O. Kreiss, J. Lorenz, “*Initial-Boundary Value Problems and the Navier-Stokes Equations*,” Academic Press, (1989).
- [24] When discussing initial-boundary value problems one sometimes demands a slightly stronger estimate that also bounds the norm of the solution at the boundary surface. For the purpose of the present article, it is sufficient to consider the weaker estimate (4) since we will show that when the determinant condition is violated, the inequality (4) cannot hold for all initial data
- [25] This condition is weaker than the uniform Kreiss condition [18] that requires that  $|\det L_-|$  must be bounded away from zero. The reason why here we do not require the uniform Kreiss condition is that it might be too strong for the case of CPBC in General Relativity. As we will see in section IV the well posed CPBC that were derived in Ref. [7] do not satisfy the uniform Kreiss condition.
- [26] Notice that the characteristic variables defined here are related with the ones  $v_{ij}^{(\pm)}$  defined in Ref. [7] according to  $u_{ij}^{(\pm)} = v_{ij}^{(\mp)} / \sqrt{2}$ .
- [27] For  $\eta = 2$  or  $\eta = 8/3$  these fields are not complete. When  $\eta = 2$  it follows that  $C_{ij}$  is traceless and as a consequence,  $\delta^{AB} V_{AB}^{(0)} = 0$ . But in this case one has the additional field  $2C + C_{xx}$  that propagates with zero speed. When  $\eta = 8/3$ ,  $\tilde{V}_{ij} = 0$  can be replaced by the fields  $V_{AB}$  and  $6V_{xA} + 5C_{xA}$ .
- [28] Actually this procedure is not unique since there is an ambiguity when first derivatives of the variables  $K_{ij}$  and  $f_{kij}$  are substituted for second derivatives of the three-metric. This ambiguity stems from the fact that one can always change the resulting expression by using the constraints  $C_{lkij} = 0$ . For definiteness, we take the choice that leads to the same boundary conditions as in Ref. [7].